Department of Physics, University of Florida — Spring 2014-15
PHZ 7608 Minicourse

**Problem Set 2: Due March 25**

1. Show that the magnification of the images of a Schwarzschild gravitational lens can be written in the form

$$\mu_\pm = \frac{1}{1-(\theta_{\rm E}/\theta_\pm)^4},$$

in which $\theta_{\rm E}$ is the Einstein angle and $\theta_\pm$ are the two solutions to the lens equation.

2. Consider a Schwarzschild gravitational lens, and a circularly symmetric source whose center is at an undeflected angle $\beta_0$ from the lens in the $x$ direction. Assume that the source has an angular diameter $2\chi$, with $\chi < \beta_0$, and model any point on the edge of the source as being on a circle described by $\beta(\phi) = (\beta_0 + \chi\cos\phi)\boldsymbol{e}_x + \chi\sin\phi\,\boldsymbol{e}_y$, with $\phi$ ranging from 0 to $2\pi$. In the limit $\chi \ll \beta_0$, show that the image is distorted into an ellipse, with a minor axis parallel to the direction of the image displacement, and with the ratio of minor to major axes given by $\beta_0/\sqrt{\beta_0^2 + 4\theta_{\rm E}^2}$.

3. Consider a gravitational-wave field $h^{\alpha\beta}$ in the far-away wave zone, satisfying the harmonic gauge condition. Prove by direct calculation that

$$R_{0j0k} = -\frac{1}{2c^2}(\mathrm{TT})^{jk}{}_{pq}\partial_{\tau\tau}h^{pq}\,.$$

4. Consider an array of particles that are able to move freely in the $x$-$y$ plane. A gravitational wave impinges on the plane in the $z$ direction. It is described by polarizations $h_+$ and $h_\times$ defined relative to the $x$-$y$-$z$ basis.

   (a) Calculate the acceleration field $\ddot{\boldsymbol{\xi}}$ experienced by the particles. Draw the lines of force in the $x$-$y$ plane when the wave is a pure + polarization, and when it is a pure $\times$ polarization. How does the pattern change when the wave is a linear superposition of each polarization?

   (b) Show that the local surface density of the particles is not affected by the gravitational wave, to first order in $h_+$ and $h_\times$. *Hint:* Evaluate the divergence of the displacement velocity field, $\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\xi}}$.

   (c) Show that the integral of the acceleration field around a closed path in the $x$-$y$ plane always vanishes. Conclude that the acceleration field can be expressed as the gradient of a potential $\Phi_{\rm GW}$,

$$\ddot{\boldsymbol{\xi}} = \boldsymbol{\nabla}\Phi_{\rm GW}\,.$$

   Determine $\Phi_{\rm GW}$ in terms of $h_+$ and $h_\times$.

5. Consider a Keplerian orbit that is circular apart from the slow decrease in radius $a$ caused by the energy lost to gravitational radiation. As a function of $\eta$, $m$, and the initial radius $a_0$, calculate the lifetime of the binary system and the number of completed orbits before the radiation reaction brings the radius to zero. Give alternative expressions for the lifetime and number of orbits in terms of $\eta$, $m$, and the initial orbital period $P$. Using these results, carry out the following estimates:

   (a) the remaining lifetime of the Hulse-Taylor binary pulsar PSR 1913+16, with $M_1 \approx M_2 \approx 1.4 M_\odot$ and $P = 7.75$ hours (assume that the orbit is circular);

   (b) the total time and number of cycles in the gravitational-wave signal from an inspiralling binary system of two $1.4 M_\odot$ compact objects, from the time it enters the LIGO-Virgo frequency band with a gravitational-wave frequency of 10 Hz to the end of the inspiral (when $a = 0$);

   (c) the remaining lifetime of the Earth-Sun system.

6. The current eccentricity of the Hulse-Taylor binary pulsar orbit is $e_0 \approx 0.6$, and its orbital period is 7.75 hours. Estimate the orbital eccentricity when gravitational waves from the system first enter the LIGO-Virgo band at 10 Hz. You may treat the eccentricity as if it were much smaller than unity when making your estimate.